# Nonlinear heat transport in a dilute gas in the presence of gravitation

Mohamed Tij
*Département de Physique, Université Moulay Ismaïl, Meknès, Morocco*

Vicente Garzó and Andrés Santos
*Departamento de Física, Universidad de Extremadura, E-06071 Badajoz, Spain*

(Received 10 July 1997)

In this paper we evaluate corrections to the Navier-Stokes constitutive equations induced by the action of a gravitational field $\mathbf{g}=-g\hat{\mathbf{z}}$ in a gas subjected to a thermal gradient parallel to a field with no convection. The analysis is performed from an exact perturbation solution of the Boltzmann equation for Maxwell molecules through second order in the field. The reference state (zeroth-order approximation) corresponds to the exact solution of the Boltzmann equation in the pure planar Fourier flow, which holds for arbitrary values of the thermal gradient. The results show that the pressure tensor becomes anisotropic, so that the momentum flux along the field direction is enhanced: $(P_zz-p)/p\approx 14.4(p^{-2}\eta^2 g\partial \ln T/\partial z)^2$. In addition, the heat flux increases (decreases) with respect to its Navier-Stokes value when the gas is heated from above (below): $q_z/q_z^{\mathrm{NS}}-1\approx 20.7(p^{-2}\eta^2 g \partial \ln T/\partial z)$. [S1063-651X(97)01812-6]

PACS number(s): 51.10.+y, 05.20.Dd, 05.60.+w, 47.50.+d

## I. INTRODUCTION

One of the most interesting and widely studied problems in fluid mechanics is the so-called Rayleigh-Bénard flow [1]. The physical situation is that of a fluid enclosed between two fixed parallel plates kept at different temperatures under the influence of a constant gravitational field perpendicular to the plates. The problem is characterized by a few dimensionless numbers. The most relevant one is the Rayleigh number

$$\mathrm{Ra}=\frac{\alpha\rho^2 c_p}{\kappa\eta} g\left(-\frac{\partial T}{\partial z}\right)L^4, \tag{1.1}$$

where $\alpha$ is the expansion coefficient, $\rho$ is the mass density, $c_p$ is the specific heat at constant pressure, $\kappa$ is the thermal conductivity, $\eta$ is the shear viscosity, $g$ is the acceleration due to gravity, $\partial T/\partial z$ is the thermal gradient (the $z$ axis is chosen orthogonal to the plates, opposite to gravity), and $L$ is the separation between the plates. One can also introduce a Froude number as

$$\mathrm{Fr}=\left(\frac{c_p T}{gL}\right)^{1/2}. \tag{1.2}$$

When the fluid is heated from below, the Rayleigh number is positive. In that situation, if Ra exceeds a critical value $\mathrm{Ra}_c\simeq 1700$, the fluid at rest becomes unstable and convection appears.

In this paper, we are interested in studying the stationary Rayleigh-Bénard flow in the absence of convection, namely, for $\mathrm{Ra}<\mathrm{Ra}_c$. In that case, the balance equations for momentum and energy imply

$$\frac{\partial}{\partial z}P_{zz}=-\rho g, \tag{1.3}$$

$$\frac{\partial}{\partial z}q_z=0, \tag{1.4}$$

where $\mathsf{P}$ is the pressure tensor, and $\mathbf{q}$ is the heat flux vector. If one assumes the applicability of the Navier-Stokes constitutive equations, namely,

$$P_{ij}^{\mathrm{NS}}=p\,\delta_{ij}, \tag{1.5}$$

$$q_z^{\mathrm{NS}}=-\kappa\frac{\partial}{\partial z}T, \tag{1.6}$$

the balance equations lead to the hydrodynamic profiles

$$\frac{\partial}{\partial z}p=-\rho g, \tag{1.7}$$

$$\frac{\partial}{\partial z}\left(\kappa\frac{\partial}{\partial z}T\right)=0. \tag{1.8}$$

This level of description is adequate for a wide range of values of the thermal gradient and the gravitational field [2]. On the other hand, if those parameters are sufficiently large, deviations from the Navier-Stokes approximations can be expected. The aim of this work is to evaluate these deviations by using kinetic theory methods.

As a prototype fluid we consider a dilute monatomic gas, which lends itself to a detailed description by means of the Boltzmann equation [3]. In a rarefied gas, the mean free path, $\lambda$, is an important distance scale parameter. Its value relative to the distance $L$ provides the Knudsen number, $\mathrm{Kn}=\lambda/L$. By using the mean free path and the thermal velocity $(k_BT/m)^{1/2}$, where $k_B$ is the Boltzmann constant and $m$ is the mass of a particle, one can define a reduced thermal gradient

$$\epsilon=\frac{\lambda}{T}\frac{\partial}{\partial z}T, \tag{1.9}$$

and a reduced gravity acceleration

 

$$g^*=\frac{\lambda}{k_BT/m}g. \tag{1.10}$$

In terms of the above quantities, the Rayleigh and Froude numbers become

$$\mathrm{Ra}=\tfrac{3}{2}(-\epsilon)g^*\mathrm{Kn}^{-4}, \tag{1.11}$$

$$\mathrm{Fr}=\left(\frac{5}{2}\frac{\mathrm{Kn}}{g^*}\right)^{1/2}, \tag{1.12}$$

where we defined $\lambda$ as $\lambda=\frac{3}{2}\eta\sqrt{k_BT/m}/p$, and took into account that in a dilute monatomic gas $\alpha=1/T$, $\kappa=\frac{3}{2}c_p\eta$, and $c_p=\frac{5}{2}k_B/m$. A necessary condition for a hydrodynamic description is $\mathrm{Kn}\ll 1$. This allows the existence of a ''bulk'' region, where the properties are rather insensitive to the details of the interactions of the particles with the boundaries. In order to obtain the first few corrections to the Navier-Stokes equations due to gravity, we will assume that $\gamma\equiv g^*\epsilon\ll 1$ (so that $\mathrm{Ra}<\mathrm{Ra}_c$), and perform a perturbation expansion in powers of $\gamma$. We will restrict ourselves to the case of Maxwell molecules, i.e., particles interacting via a potential $V(r)\propto r^{-4}$. The reason is twofold. First, the velocity moments of the nonlinear Boltzmann collision operator can be expressed as combinations of the moments of the distribution function [3]. Second, the Boltzmann equation admits an exact solution for the pure Fourier flow (i.e., in the absence of gravity) with *arbitrary* thermal gradients [4]. In this solution, the Navier-Stokes equations are exactly verified, even for large values of $\epsilon$. As a consequence, deviations associated with $\gamma\neq 0$ are only due to the presence of gravity.

## II. DESCRIPTION OF THE PROBLEM FROM THE BOLTZMANN EQUATION

Let us consider a dilute gas described by the Boltzmann equation [3]

$$\frac{\partial}{\partial t}f+\mathbf{v}\cdot\nabla f+\frac{\mathbf{F}}{m}\cdot\frac{\partial}{\partial\mathbf{v}}f=J[f,f]. \tag{2.1}$$

Here, $f(\mathbf{r},\mathbf{v},t)$ is the one-particle distribution function, $\mathbf{F}$ is an external force, and $J[f,f]$ is the nonlinear Boltzmann collision operator. The densities of the conserved quantities (mass, momentum, and energy), as well as their fluxes, are given as the first few velocity moments of $f$. In particular,

$$n=\int d\mathbf{v}\,f \tag{2.2}$$

is the local density,

$$\mathbf{u}=\frac{1}{n}\int d\mathbf{v}\,\mathbf{v}f \tag{2.3}$$

is the local flow velocity,

$$T=\frac{m}{3nk_B}\int d\mathbf{v}(\mathbf{v}-\mathbf{u})^2f \tag{2.4}$$

is the local temperature,

$$\mathsf{P}=m\int d\mathbf{v}(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})f, \tag{2.5}$$

is the pressure tensor, and

$$\mathbf{q}=\frac{m}{2}\int d\mathbf{v}(\mathbf{v}-\mathbf{u})^2(\mathbf{v}-\mathbf{u})f \tag{2.6}$$

is the heat flux vector.

We are interested in a stationary state with spatial variation only along a given direction (say $z$), and a constant external field $\mathbf{F}=-mg\hat{\mathbf{z}}$ along that direction. In addition, as stated in Sec. I, we assume that there is no convection, i.e., $\mathbf{u}=\mathbf{0}$, and that the particles interact through the Maxwell potential. For this interaction, the collision rate appearing in the collision operator is independent of the velocity. For the sake of clarity, let us introduce dimensionless quantities. To do so, we choose an *arbitrary* point $z_0$ in the bulk region, and take the quantities at that point (denoted by a subscript 0) as reference units. Therefore, we define $T^*\equiv T/T_0$, $p^*\equiv p/p_0$, $\mathbf{v}^*\equiv\mathbf{v}/v_0$, $f^*\equiv n_0^{-1}v_0^3f$, and $g^*\equiv g\lambda_0/v_0^2$, where $p=nk_BT$ is the hydrostatic pressure, and $v_0\equiv(k_BT_0/m)^{1/2}$ is a thermal velocity. In the case of the spatial variable $z$, it is convenient to rescale it in a nonlinear way that takes into account the local dependence of the density. Consequently, we define

$$s=\frac{1}{n_0\lambda_0}\int_{z_0}^{z}dz'\,n(z'). \tag{2.7}$$

Under the above conditions, Eq. (2.1) becomes

$$\left(v_z^*\partial_s-g^*\frac{T^*}{p^*}\frac{\partial}{\partial v_z^*}\right)f^*=\frac{n_0\lambda_0}{v_0}\frac{T^*}{p^*}J[f^*,f^*], \tag{2.8}$$

where $\partial_s\equiv\partial/\partial s$. In Eq. (2.8), we made use of the property $J[f,f]=n_0^2v_0^{-3}J[f^*,f^*]$, which only holds for Maxwell molecules.

According to the geometry of the problem, the relevant velocity moments are defined as

$$M_{\alpha\beta}=\int d\mathbf{v}^*v^{*2\alpha}v_z^{*\beta}f^*. \tag{2.9}$$

In particular, $M_{10}=3p^*$ and $M_{00}=p^*/T^*$. The Boltzmann equation (2.8) is then formally equivalent to the following hierarchy of moment equations:

$$\partial_sM_{\alpha,\beta+1}+g^*\frac{T^*}{p^*}(\beta M_{\alpha,\beta-1}+2\alpha M_{\alpha-1,\beta+1})=J_{\alpha\beta}, \tag{2.10}$$

where

$$J_{\alpha\beta}=\frac{n_0\lambda_0}{v_0}\frac{T^*}{p^*}\int d\mathbf{v}^*v^{*2\alpha}v_z^{*\beta}J[f^*,f^*]. \tag{2.11}$$

In the case of Maxwell molecules, the collisional moments $J_{\alpha\beta}$ are bilinear combinations of moments $M_{\alpha'\beta'}$ of degree

$2\alpha'+\beta'$ less than or equal to $2\alpha+\beta$. Through fourth degree, their explicit expressions are [5–7] $J_{00}=J_{01}=J_{10}=0$, and

$$J_{02}=-\tfrac{3}{2}(M_{02}-p^*), \tag{2.12}$$

$$J_{11}=-M_{11}, \tag{2.13}$$

$$J_{03}=-\tfrac{9}{4}\left(M_{03}-\tfrac{1}{3}M_{11}\right), \tag{2.14}$$

$$J_{20}=-M_{20}-\frac{T^*}{p^*}\left[\frac{3}{2}(M_{02}-p^*)^2-15p^{*2}\right], \tag{2.15}$$

$$J_{12}=-\tfrac{7}{4}M_{12}+\tfrac{1}{4}M_{20}-\frac{T^*}{p^*}\left(M_{02}^2-\tfrac{15}{4}p^*M_{02}-\tfrac{9}{4}p^{*2}\right), \tag{2.16}$$

$$J_{04}=-\omega M_{04}+\tfrac{3}{14}(4\omega-7)M_{12}+\tfrac{3}{70}(7-2\omega)M_{20} -\frac{3}{70}\frac{T^*}{p^*}[4(9\omega-22)M_{02}^2-3(24\omega-47)p^*M_{02} -3(41-12\omega)p^{*2}], \tag{2.17}$$

where $\omega\simeq 2.8097$, and we have taken into account the symmetries of the problem.

Equation (2.8) admits as a trivial solution the *equilibrium* state characterized by $T^*=1$ and $p^*=1-g^*s$. The latter equation is nothing but the barometric formula $p(z)=p_0\exp[-mg(z-z_0)/k_BT]$. On the other hand, in the absence of gravitation ($g=0$), Eq. (2.8) has an *exact* solution [4,8] characterized by a constant pressure, $p^*=1$, and a ''linear'' temperature profile

$$T^*=1+\epsilon_0 s, \tag{2.18}$$

where $\epsilon_0$ is a constant. On the other hand, the reduced thermal gradient defined in Eq. (1.9) is a local quantity, namely, $\epsilon(z)=\epsilon_0\sqrt{T_0/T(z)}$, where we have taken into account that $\lambda\propto n^{-1}T^{1/2}$ for Maxwell molecules. It must be remarked that the solution applies to arbitrary values of $\epsilon_0$. The velocity moments $M_{\alpha\beta}$ are *polynomials* in $s$ of degree $\alpha+I(\beta/2)-1$, where $I()$ denotes the integer part, except $M_{00}=(1+\epsilon_0 s)^{-1}$. Their explicit expressions for $2\leqslant 2\alpha+\beta\leqslant 5$ are [6–8]

$$M_{10}=3,\quad M_{02}=1, \tag{2.19a}$$

$$M_{11}=-5\epsilon_0,\quad M_{03}=-3\epsilon_0, \tag{2.19b}$$

$$M_{20}=15(1+\epsilon_0 s)+70\epsilon_0^2,\quad M_{12}=5(1+\epsilon_0 s)+34\epsilon_0^2,$$

$$M_{04}=3(1+\epsilon_0 s)+\frac{162}{7}\epsilon_0^2, \tag{2.19c}$$

$$M_{21}=-70(1+\epsilon_0 s)\epsilon_0-\left(\frac{4652}{9}+\frac{112}{\omega'}\right)\epsilon_0^3, \tag{2.19d}$$

$$M_{13}=-42(1+\epsilon_0 s)\epsilon_0-\left[\frac{4652}{15}+\frac{336}{5\omega'}+\frac{216}{5\omega''}\left(\frac{8}{7}+\frac{1}{\omega'}\right)\right]\epsilon_0^3, \tag{2.19e}$$

$$M_{05}=-30(1+\epsilon_0 s)\epsilon_0-\left[\frac{4652}{21}+\frac{48}{\omega'}+\frac{48}{\omega''}\left(\frac{8}{7}+\frac{1}{\omega'}\right)\right]\epsilon_0^3, \tag{2.19f}$$

where $\omega'\simeq 2.0133$ and $\omega''\simeq 2.3555$. The fact that $M_{11}=-5\epsilon_0$ means that the Fourier law, Eq. (1.6), holds even for large thermal gradients. A similar conclusion is obtained from an exact solution of the Bhatnagar-Gross-Krook (BGK) model for general interactions [9]. Notice that Eq. (2.18) leads to $(\partial/\partial z)^2T^2=0$, which is consistent with Eq. (1.8). The nonlinear relationship between $s$ and $z$ is, according to Eq. (2.7), $s=s^{(0)}(z)$, where

$$s^{(0)}(z)=\epsilon_0^{-1}\left[\left(1+2\frac{\epsilon_0}{\lambda_0}(z-z_0)\right)^{1/2}-1\right]. \tag{2.20}$$

As stated in Sec. I, the motivation for this paper was to analyze the influence of gravitation on the profiles and transport properties of the above steady Fourier flow. The presence of the term proportional to $g^*$ in Eq. (2.10) complicates the solution of the hierarchy enormously, since the moments are no longer just polynomials in $s$. However, from a practical point of view the value of the gravity acceleration is sufficiently small as to justify a perturbation analysis. More specifically, we will carry out a perturbation expansion in powers of $\gamma\equiv\epsilon_0 g^*$:

$$f^*=f^{(0)}+f^{(1)}\gamma+f^{(2)}\gamma^2+\cdots, \tag{2.21}$$

where the reference state $f^{(0)}$ represents the pure steady Fourier flow corresponding to the actual values of pressure, temperature, and thermal gradient at the point of interest $z=z_0$. Analogously,

$$M_{\alpha\beta}=M_{\alpha\beta}^{(0)}+M_{\alpha\beta}^{(1)}\gamma+M_{\alpha\beta}^{(2)}\gamma^2+\cdots, \tag{2.22}$$

$$p^*=p^{(0)}+p^{(1)}\gamma+p^{(2)}\gamma^2+\cdots, \tag{2.23}$$

$$T^*=T^{(0)}+T^{(1)}\gamma+T^{(2)}\gamma^2+\cdots, \tag{2.24}$$

where the first few moments $M_{\alpha\beta}^{(0)}$ are given by Eqs. (2.19), $p^{(0)}=1$, and $T^{(0)}=1+\epsilon_0 s$. By definition, $p^{(k)}(0)=T^{(k)}(0)=\partial T^{(k)}/\partial s|_{s=0}=0$ for $k\geqslant 1$. It must be emphasized that the terms of order $\gamma^k$ are *nonlinear* functions of $\epsilon_0$, and no restriction as to the order on $\epsilon_0$ exists.

## III. PERTURBATION EXPANSION

In this section we obtain the hydrodynamic profiles $p^{(k)}$ and $T^{(k)}$, and the fluxes $M_{02}^{(k)}$ and $M_{11}^{(k)}$ through order $k=2$. Inserting Eqs. (2.22)–(2.24) into Eq. (2.10), one obtains

$$\partial_s M_{\alpha,\beta+1}^{(k)}+\frac{1}{\epsilon_0}\sum_{k'=0}^{k-1}\left(\frac{T^*}{p^*}\right)^{(k')}(\beta M_{\alpha,\beta-1}^{(k-1-k')}+2\alpha M_{\alpha-1,\beta+1}^{(k-1-k')}) =J_{\alpha\beta}^{(k)}. \tag{3.1}$$

In particular,

$$\partial_s M^{(1)}_{\alpha,\beta+1}+\frac{1}{\epsilon_0}(1+\epsilon_0 s)(\beta M^{(0)}_{\alpha,\beta-1}+2\alpha M^{(0)}_{\alpha-1,\beta+1})=J^{(1)}_{\alpha\beta}, \tag{3.2}$$

$$\partial_s M^{(2)}_{\alpha,\beta+1}+\frac{1}{\epsilon_0}\{(1+\epsilon_0 s)(\beta M^{(1)}_{\alpha,\beta-1}+2\alpha M^{(1)}_{\alpha-1,\beta+1}) + [T^{(1)}-(1+\epsilon_0 s)p^{(1)}](\beta M^{(0)}_{\alpha,\beta-1}+2\alpha M^{(0)}_{\alpha-1,\beta+1})\} = J^{(2)}_{\alpha\beta}. \tag{3.3}$$

In these equations, one needs to take into account that $M^{(1)}_{00}=(T^{(0)}p^{(1)}-T^{(1)})/T^{(0)2}$, $M^{(2)}_{00}=[T^{(1)2}-T^{(0)}(T^{(1)}p^{(1)}+T^{(2)})]/T^{(0)3}$, and $M^{(k)}_{10}=3p^{(k)}$. Inspection of these equations shows that the polynomial structure of the solution corresponding to the pure Fourier flow is extended to the solution of order $k$. More specifically, $T^{(k)}$ is a polynomial in $s$ of degree $k+1$, and $M^{(k)}_{\alpha\beta}$, $2\alpha+\beta\geqslant 2$, is a polynomial of degree $\alpha+I(\beta/2)+k-1$:

$$T^{(k)}(s)=\sum_{\ell=0}^{k-1} T^{(k,\ell)}s^{k+1-\ell}, \tag{3.4}$$

$$M^{(k)}_{\alpha\beta}=\sum_{\ell=0}^{\alpha+I(\beta/2)+k-1} \mu^{(k,\ell)}_{\alpha\beta}s^{\alpha+I(\beta/2)+k-1-\ell}, \tag{3.5}$$

where the coefficients are so far unknown. According to Eqs. (3.4) and (3.5), the left-hand side of Eq. (3.1) is a polynomial of degree $\alpha+I[(\beta-1)/2]+k-1$, while the right-hand side is a polynomial of degree $\alpha+I(\beta/2)+k-1$. Consequently, if $\beta=$even, the coefficient of $\alpha+\beta/2+k-1$ on the right-hand side of Eq. (3.1) must vanish, and this allows one to obtain the coefficients $\mu^{(k,0)}_{\alpha'\beta'}$ for $2\alpha'+\beta'=2\alpha+\beta$. The general solution scheme then proceeds as follows:

$$\{\mu^{(k,\ell)}_{d}\}\to\{\mu^{(k,\ell)}_{d-1}\}\to\{\mu^{(k,\ell+1)}_{d-2}\}\to\{\mu^{(k,\ell+1)}_{d-3}\}\to\cdots, \tag{3.6}$$

where $\{\mu^{(k,\ell)}_{d}\}$ denotes the set of coefficients $\{\mu^{(k,\ell)}_{\alpha\beta};2\alpha+\beta=d\}$, and $d$ is even in the first set of Eq. (3.6). Consequently, in order to determine completely $M^{(k)}_{\alpha\beta}$ one needs to make use of the collisional moments $J_{\alpha'\beta'}$ with $2\alpha'+\beta'\leqslant 2(2\alpha+\beta+k-1)$. As a matter of fact, to obtain the heat flux to second order, one needs to know the collisional moments through the eighth degree. Since, to the best of our knowledge, only the collisional moments through the fifth degree are given in the literature [5], here we have used a recent evaluation of moments of higher degree [10].

Let us first consider the hierarchy (3.2). Making $(\alpha,\beta)=(0,1)$, one obtains $\mu^{(1,0)}_{02}=-1/\epsilon_0$, which is equivalent to Eq. (1.3). Next, if $(\alpha,\beta)=(1,0)$, one obtains $\mu^{(1,0)}_{11}=0$, which is equivalent to Eq. (1.4). The first nontrivial result is obtained by making $(\alpha,\beta)=(0,2)$. In this case, one has $\mu^{(1,0)}_{10}=3\mu^{(1,0)}_{02}=-3/\epsilon_0$ and

$$\mu^{(1,1)}_{02}=-\tfrac{2}{3}\mu^{(1,0)}_{03}, \tag{3.7}$$

where we have taken into account that $\mu^{(k,k)}_{10}=0$. Now we take $(\alpha,\beta)=(1,1),(0,3)$, which yields

$$\mu^{(1,0)}_{12}=-\tfrac{5}{2}, \tag{3.8}$$

$$\mu^{(1,0)}_{03}=-\tfrac{4}{3}\left(1+\tfrac{2}{3}\mu^{(1,0)}_{04}\right). \tag{3.9}$$

Next we take $(\alpha,\beta)=(2,0)$, (1,2), and (0,4):

$$T^{(1,0)}-\tfrac{1}{15}\mu^{(1,0)}_{20}-1=0, \tag{3.10}$$

$$T^{(1,0)}+\tfrac{1}{20}\mu^{(1,0)}_{20}-\tfrac{1}{8}=0, \tag{3.11}$$

$$T^{(1,0)}+\frac{7-2\omega}{70}\mu^{(1,0)}_{20}-\frac{\omega}{3}\mu^{(1,0)}_{04}-\frac{20\omega-7}{28}=0. \tag{3.12}$$

The solution is $T^{(1,0)}=\frac{1}{2}$, $\mu^{(1,0)}_{20}=-\frac{15}{2}$, and $\mu^{(1,0)}_{04}=-\frac{3}{2}$. Substitution into Eqs. (3.9) and (3.7) gives $\mu^{(1,0)}_{03}=\mu^{(1,1)}_{02}=0$. This completes the determination of the second-degree moments to first order.

To determine the heat flux to first order, one needs explicit knowledge of the sixth-degree collisional moments. Furthermore, the eighth-degree collisional moments are needed in the evaluation of the heat flux to second order. The algebra is straightforward but rather tedious and here we quote only the final results:

$$p^*=1-\frac{1}{\epsilon_0}s\gamma+O(\gamma^3), \tag{3.13}$$

$$M_{02}=1-\frac{1}{\epsilon_0}s\gamma+\frac{128}{45}\gamma^2+\mathcal{O}(\gamma^3), \tag{3.14}$$

$$T^*=1+\epsilon_0 s+\tfrac{1}{2}s^2\gamma-s^2\left(\frac{468}{45}-\frac{1}{3\epsilon_0}s\right)\gamma^2+\mathcal{O}(\gamma^3), \tag{3.15}$$

$$M_{11}=-5\epsilon_0\left[1+\frac{46}{5}\gamma+\left(\frac{12}{5\epsilon_0^2}+503.7\right)\gamma^2+\mathcal{O}(\gamma^3)\right], \tag{3.16}$$

$$M_{03}=-3\epsilon_0\left[1+\frac{206}{21}\gamma+\left(\frac{164}{63\epsilon_0^2}+\frac{64}{45\epsilon_0}s+550.3\right)\gamma^2+\mathcal{O}(\gamma^3)\right]. \tag{3.17}$$

Equation (3.13) shows that Eq. (1.7) is still valid to second order. Nevertheless, $P_{ij}\neq P^{\rm NS}_{ij}=p\delta_{ij}$ to that order. More specifically, Eq. (3.14) implies that

$$\frac{P_{zz}-p}{p}=\frac{128}{45}\gamma^2+\mathcal{O}(\gamma^3). \tag{3.18}$$

Although we have used the space variable $s$ as mathematically convenient, let us go back to the actual space coordinate $z$. The nonlinear relationship between $s$ and $z$ can be obtained from Eqs. (2.7), (3.13), and (3.15):

$$s(z)=s^{(0)}(z)+s^{(1)}(z)\gamma+s^{(2)}(z)\gamma^2+\mathcal{O}(\gamma^3), \tag{3.19}$$

where $s^{(0)}(z)$ is given by Eq. (2.20), and

$$s^{(1)}(z)=-\frac{1}{2\epsilon_0}s^{(0)2},\quad s^{(2)}(z)=\frac{104\epsilon_0^2+5\epsilon_0 s^{(0)}+5}{30\epsilon_0^2(1+\epsilon_0 s^{(0)})}s^{(0)3}. \tag{3.20}$$

Substituting Eq. (3.19) into Eq. (3.15), we obtain the temperature profile

$$T(z)=T_0\left\{\left[1+2\frac{\epsilon_0}{\lambda_0}(z-z_0)\right]^{1/2} - \frac{104}{15}\frac{\left[1+2\frac{\epsilon_0}{\lambda_0}(z-z_0)\right]^{3/2}-1-3\frac{\epsilon_0}{\lambda_0}(z-z_0)}{\epsilon_0^2\left[1+2\frac{\epsilon_0}{\lambda_0}(z-z_0)\right]^{1/2}}\gamma^2 + \mathcal{O}(\gamma^3)\right\}. \tag{3.21}$$

Thus, the deviation from the profile given by Eq. (1.8) is of second order, namely,

$$\frac{\partial}{\partial z}\left(\kappa\frac{\partial}{\partial z}T\right)\Bigg|_{z=z_0}=-\frac{104}{5}\frac{\kappa_0 T_0}{\lambda_0^2}\gamma^2+\mathcal{O}(\gamma^3). \tag{3.22}$$

On the other hand, since $M_{11}/(-5\epsilon_0)=q_z/q_z^{\mathrm{NS}}$, Eq. (3.16) shows that the correction to the Fourier law, Eq. (1.6), is of first order. The results predict that when one heats from above (i.e., $\epsilon_0>0$, so that $\gamma>0$), the gravitational field produces an enhancement of the heat flux with respect to its Navier-Stokes value; the opposite effect occurs when one heats from below, at least for $|\gamma|\ll 1$.

Following the same procedure, one might obtain higher correction terms. However, not only the algebra involved becomes more and more complicated, but its applicability may be limited by the possible asymptotic character of the series. For illustrative purposes, it is useful to consider Padé approximants [11]. In Fig. 1 we plot the ratio $q_z/q_z^{\mathrm{NS}}$, taking the Padé approximants (1,1) and (0,2) of Eq. (3.16), at $g^*=0.01$ in the range $-0.015\leqslant\gamma\leqslant 0.015$. There is a region ($|\gamma|\lesssim 0.01$) where both curves practically overlap. This allows us to estimate that at $g^*=0.01$ the heat flux increases by a 12% with respect to its Navier-Stokes value if one heats from above with $\epsilon=1$, while it decreases by a 7% if one heats from below with $\epsilon=-1$.

## IV. DISCUSSION

In this paper we have investigated the influence of gravity on the heat transport across a fluid in a slab, in the absence of convection. This means a restriction to values of the Rayleigh number less than the critical value, $\mathrm{Ra}<\mathrm{Ra}_c\simeq 1700$. Usually, one adopts a hydrodynamic description in the sense that $g$ explicitly appears in the balance equations, but it is assumed that the constitutive relations between fluxes and gradients are those of Navier-Stokes; thus only the hydrodynamic profiles are affected by gravity. Nevertheless, as a matter of principle, a certain deviation from the Navier-Stokes equations can be expected. The evaluation of such a deviation in a dilute gas of Maxwell molecules was the main motivation of this paper.

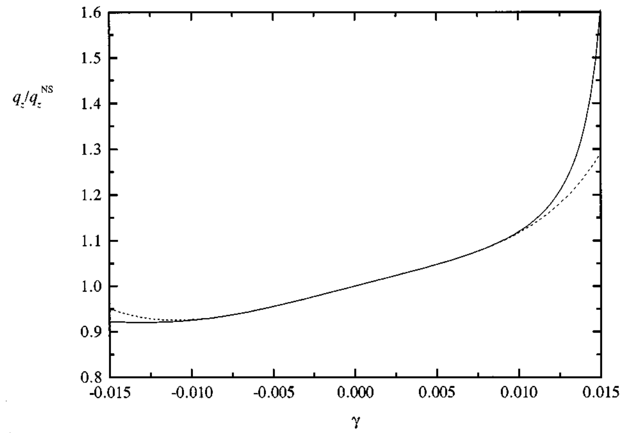


FIG. 1. Plot of the ratio $q_z/q_z^{\mathrm{NS}}$ as a function of $\gamma$ at $g^*=0.01$. The solid and dashed lines are the Padé approximants (1,1) and (0,2), respectively, derived from Eq. (3.16).

We have solved the steady nonlinear Boltzmann equation by means of a perturbation expansion around the pure Fourier flow state (i.e., $g=0$). In the latter state, the Navier-Stokes equations are *exact*, even for arbitrary values of the thermal gradient [4]. Consequently, the deviations found are exclusively due to the action of gravity. The main results are summarized by Eqs. (3.16), (3.18), and (3.21). While the anisotropy of the pressure tensor and the correction to the temperature profile are of second order, the correction to the heat flux is of first order, so that the latter depends on the sign of the thermal gradient. This implies an inhibition (enhancement) of the heat transport when the gas is heated from below (above).

Although our results have been obtained for Maxwell molecules, we expect that most of them can be extended to other interaction potentials when the proper temperature dependence of the thermal conductivity is taken into account. For instance, Eqs. (3.16), (3.18), and (3.22) can still hold, except for a change in the numerical coefficients. This expectation is based on the fact that Monte Carlo simulations of the Boltzmann equation for hard spheres have confirmed the reliability of the solutions for Maxwell molecules in cases such as the shear flow [12], the pure Fourier flow [13], and the Poiseuille flow [14].

Finally, it is obvious that the effects analyzed here are practically irrelevant for gases under terrestrial conditions (for instance, in the case of air at room temperature, $g^*\sim 10^{-11}$). The same can be said of recent numerical solutions of the Boltzmann equation showing the existence of the Rayleigh-Bénard instability in rarefied gases [15]. Nevertheless, apart from its theoretical interest, the issue addressed in this paper may be useful in more complex systems, such as viscous liquids or low-density granular media.

## ACKNOWLEDGMENTS

V.G. and A.S. acknowledge partial support from the DGICYT (Spain) through Grant No. PB94-1021. M.T. is grateful to the University of Extremadura at Badajoz for its hospitality during a visit.

[1] D. J. Tritton, *Physical Fluid Dynamics* (Oxford University Press, Oxford, 1988).
[2] M. Mareschal, M. Malek Mansour, A. Puhl, and E. Kestemont, Phys. Rev. Lett. **61**, 2550 (1988); A. Puhl, M. Malek Mansour, and M. Mareschal, Phys. Rev. A **40**, 1999 (1989); D. Risso and P. Cordero, J. Stat. Phys. **82**, 1453 (1996).
[3] J. A. McLennan, *Introduction to Nonequilibrium Statistical Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1989).
[4] E. S. Asmolov, N. K. Makashev, and V. I. Nosik, Dokl. Akad. Nauk. **249**, 577 (1979) [Sov. Phys. Dokl. **24**, 892 (1979)].
[5] C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (Academic, New York, 1980).
[6] A. Santos and V. Garzó, Physica A **213**, 409 (1995).
[7] M. Tij and A. Santos, Phys. Fluids **7**, 2858 (1995).
[8] A. Santos and V. Garzó, in *Rarefied Gas Dynamics 19*, edited by J. Harvey and G. Lord (Oxford University Press, Oxford, 1995), Vol. 1, pp. 13–22.
[9] A. Santos, J. J. Brey, and V. Garzó, Phys. Rev. A **34**, 5047 (1986); A. Santos, J. J. Brey, C. S. Kim, and J. W. Dufty, *ibid.* **39**, 320 (1989); C. S. Kim, J. W. Dufty, A. Santos, and J. J. Brey, *ibid.* **39**, 328 (1989).
[10] M. Tij and M. Sebanne (unpublished).
[11] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, Singapore, 1987).
[12] J. Gómez Ordóñez, J. J. Brey, and A. Santos, Phys. Rev. A **39**, 3038 (1989); **41**, 810 (1990).
[13] J. M. Montanero, M. Alaoui, A. Santos, and V. Garzó, Phys. Rev. E **49**, 367 (1993).
[14] M. Malek Mansour, F. Baras, and A. L. Garcia, Physica A **240**, 255 (1997).
[15] C. Cercignani and S. Stefanov, Transp. Theory Stat. Phys. **21**, 371 (1992); Y. Sone, K. Aoki, H. Sugimoto, and H. Motohashi, in *Rarefied Gas Dynamics 19* (Ref. [8]), Vol. 1, pp. 135–141.